Emittance of Semi-Infinite Absorbing and Isotropically Scattering Medium with Refractive Index Greater than Unity

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An approximate solution is presented for the integrodifferential equation that governs the radiant intensity and the source function within a semi-infinite absorbing-scattering medium with an index of refraction different from unity. The exponential kernel substitution has been used to develop a closed form analytical expression for the directional emittance as a function of the refractive index and the scattering albedo. The scattering has been considered as isotropic, and the Fresnel relations have been used to evaluate the reflection and the transmission at the interface. The directional and the hemispherical emittances are presented for a refractive index range between 1 and 2 and for scattering albedo between 0 and 1. The results indicate that the directional and hemispherical emittances of a medium with a refractive index larger than unity can be either smaller, because of interface reflection, or larger, because of scattering, than that associated with a medium of a unity index of refraction. A generalized graph is presented from which an approximate value for the directional emittance can be obtained for a wide range of optical conditions. Based on other applications in the field of radiative transfer with similar geometry and on comparison, when possible, with exact solutions, the approximate method which has been used should be accurate in predicting directional and hemispherical emittances to within 10%.

Introduction

THE analysis of radiant transport within and through a medium which absorbs and scatters energy has been a problem of great interest for many years. Most of the analyses however have dealt mainly with a medium having an index of refraction equal to unity. This restriction simplifies and decreases the computational effort for the solution of the transport equation but neglects the contributions of the interface reflection and transmission. In many cases, such as radiant energy transfer through liquids, glass, and plastics, the above assumption of no reflection at the interface is not valid. Several investigators[1-7] have considered the influence of the refractive index on radiant transport and have demonstrated that it plays an important role.

A semi-infinite absorbing and isotropically scattering medium with a unity refractive index has been treated in detail by Chandrasekhar.[8] In addition, Edwards and Bobco[9] used a method similar to the one applied in this study to obtain an approximate solution to the above problem. Turner and Love,[5] by using the Monto Carlo method, examined the effect of the refractive index and the scattering albedo on the emission from a slab. Their method has the disadvantage of being lengthy and relatively complicated. The present investigators have considered the simple case of emittance from a semi-infinite absorbing and isotropically scattering medium with a refractive index different than unity in an effort to exhibit more clearly the role played by the interface in such a problem. The exponential kernel substitution has been used to obtain an approximate but simple close form solution for the governing transport equation. A closed form analytical expression is presented for the directional emittance as a function of the refractive index and the scattering albedo.

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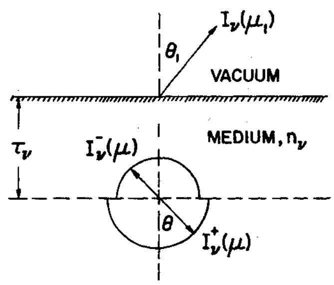

Fig. 1 Schematic of coordinate system.

Formulation

The system chosen for this study is a semi-infinite planar medium at a uniform temperature which scatters isotropically and is characterized by a spectral scattering albedo ω_ν (ratio of scatter σ_ν to extinction β_ν coefficient) and a refractive index n_ν. The subscript ν refers to the frequency under consideration. The medium is not subjected to external radiation and is bounded by a medium with a refractive index of unity, Fig. 1. The radiant intensity within the medium is governed by the transport equation which can be stated for the above problem by

$$\mu\, dI_\nu(\tau_\nu, \mu)/d\tau_\nu + I_\nu(\tau_\nu, \mu) = S_\nu(\tau_\nu) \tag{1}$$

where I_ν is the intensity,

$$\tau_\nu \text{ is the optical depth } \left(\tau_\nu = \int_0^x \beta_\nu\, dx\right)$$

$\mu = \cos\theta$, and S_ν is the source function which is defined by

$$S_\nu(\tau_\nu) = (1-\omega_\nu)n_\nu{}^2 I_{b_\nu}(T) + \frac{\omega_\nu}{2}\int_{-1}^{1} I_\nu(\tau_\nu, \mu)\, d\mu \tag{2}$$

The solution to this transport equation for the semi-infinite medium described above could be stated in terms of the intensities in the plus and the minus direction by

$$I_\nu{}^+(\tau_\nu, \mu) = I_\nu{}^+(0, \mu)\exp\left(\frac{-\tau_\nu}{\mu}\right) + \int_0^{\tau_\nu} S_\nu(t)\exp\left[\frac{-(\tau_\nu - t)}{\mu}\right]\frac{dt}{\mu} \tag{3}$$

and

$$I_\nu^-(\tau_\nu, \mu) = \int_{\tau_\nu}^{\infty} S_\nu(t) \exp\left[\frac{-(t-\tau_\nu)}{\mu}\right] \frac{dt}{\mu} \tag{4}$$

The plus and the minus superscripts indicate the directions as shown in Fig. 1. The source function S_ν is then given by

$$S_\nu(\tau_\nu) = (1-\omega_\nu) n_\nu^2 I_{b_\nu}(T) + \frac{\omega_\nu}{2} \times \left[\int_0^1 I_\nu^+(0,\mu) \exp\left(\frac{-\tau_\nu}{\mu}\right) d\mu + \int_0^\infty S_\nu(t) E_1(|\tau_\nu - t|)\, dt\right] \tag{5}$$

where $I_{b_\nu}(T)$ is black body intensity corresponding to the temperature of the medium and $E_1(t)$ is the exponential integral defined by

$$E_1(t) = \int_0^1 \exp\left(\frac{-t}{\mu}\right) \frac{d\mu}{\mu} \tag{6}$$

The intensities at the interface $I_\nu^+(0, \mu)$ and $I_\nu^-(0, \mu)$ can be determined from the final solution of Eqs. (3) and (4). Because there is no external incidence at the interface, the intensity $I_\nu^+(0, \mu)$ is only the result of back reflection at the interface of the medium and is given by

$$I_\nu^+(0,\mu) = \rho_\nu(\mu)\, I_\nu^-(0,\mu) = \rho_\nu(\mu) \int_0^\infty S_\nu(t) \exp\left(\frac{-t}{\mu}\right) \frac{dt}{\mu} \tag{7}$$

where $\rho_\nu(\mu)$ is the interface reflectance. The interface is assumed to be smooth, and its reflection and transmission characteristics are governed by Snell's law and the Fresnel relations. The complex part of the index of refraction for the medium has been assumed to be small in comparison with the real part such that the interface reflection and refraction can be treated as if the medium was pure dielectric. Such an assumption is reasonable for glass, plastics, and liquids in a wide region of the spectrum. It is important to note that the reflection $\rho_\nu(\mu)$ in Eq. (7) equals unity for all angles larger than the critical angle. By imposing the preceding assumptions, the reflectance at the interface can be calculated by using[10]

$$\rho_\nu(\mu) = \frac{1}{2}\left[\frac{\{n_\nu \mu - [1 + n_\nu^2(1-\mu^2)]^{1/2}\}^2}{\{n_\nu \mu + [1 - n_\nu^2(1-\mu^2)]^{1/2}\}^2} + \frac{\{\mu - n_\nu[1 - n_\nu^2(1-\mu^2)]^{1/2}\}^2}{\{\mu + n_\nu[1 - n_\nu^2(1-\mu^2)]^{1/2}\}^2}\right] \tag{8}$$

and the critical angle is defined by

$$\mu_c = [(n_\nu^2 - 1)/n_\nu^2]^{1/2} \tag{9}$$

The directional emittance at $\mu_1 = \cos\theta_1$, as measured on the outside side of the interface (Fig. 1), is defined by

$$\varepsilon_\nu(\mu_1) = I_\nu(\mu_1)/I_{b_\nu}(T) \tag{10}$$

where $I_\nu(\mu_1)$ is the intensity at the vacuum side of the interface in the direction of μ_1. The intensity in that direction is the result of an interface transmission and is given by

$$I_\nu(\mu_1) = (1/n_\nu^2)\{1 - \rho_\nu([1-(1-\mu_1^2)/n_\nu^2]^{1/2})\} \times I_\nu^-(0, [1-(1-\mu_1^2)/n_\nu^2]^{1/2}) \tag{11}$$

The square root in Eq. (11) represents the direction in which the intensity within the medium must hit the interface in order to appear at the vacuum side in the direction of μ_1 (Snell's law). By combining Eqs. (4, 10, and 11), the directional emittance can be expressed by

$$\varepsilon_\nu(\mu_1) = \frac{\{1 - \rho_\nu([1-(1-\mu_1^2)/n_\nu^2]^{1/2})\}}{n_\nu^2 I_{b_\nu}(T)[1-(1-\mu_1^2)/n_\nu^2]^{1/2}} \times \int_0^\infty S_\nu(t) \exp\left\{\frac{-t}{[1-(1-\mu_1^2)/n_\nu^2]^{1/2}}\right\} dt \tag{12}$$

and the hemispherical emittance is given by

$$\varepsilon_{H_\nu} = 2\int_0^1 \varepsilon_\nu(\mu_1)\mu_1\, d\mu_1 \tag{13}$$

Kernel Substitution

It is apparent from Eq. (12) that the solution for the source function is the essential step for determining the directional emittance. The exact numerical solution of the integral equation governing the source function is lengthy and complicated. The present study examines an approximate solution to the governing equation which can be rearranged by combining Eqs. (5) and (7) to define a dimensionless source function which is given by

$$\phi_\nu(\tau_\nu) = 1 + \frac{\omega_\nu}{2}\int_0^\infty \phi_\nu(t)\{E_1(|\tau_\nu - t|) + E_1(\tau_\nu + t) - B_n(\tau_\nu + t)\}\, dt \tag{14}$$

where

$$\phi_\nu(\tau_\nu) = S_\nu(\tau_\nu)/n_\nu^2(1-\omega_\nu)I_{b_\nu}(T) \tag{15}$$

and

$$B_n(\tau) = \int_0^1 [1 - \rho_\nu(\mu)] \exp\left(\frac{-t}{\mu}\right)\frac{d\mu}{\mu} \tag{16}$$

The exponential kernel substitution is used in order to obtain an approximate solution to the above integral equation. The procedure consists first of replacing each of the terms in the kernel by an exponential term which is approximately equivalent to it. The integral equation is then transformed into a linear ordinary differential equation with constant coefficients which can be easily solved. Such an approximation has been applied in radiative transfer studies[11] and has been found to be valid, especially in the thick limit.

The last term in the kernel has been evaluated[12] for different values of the index of refraction and has been found to be well represented by an exponential approximation equivalent to

$$B_n(t) = a_n \exp(-b_n t) \tag{17}$$

where a_n and b_n are constants but are dependent on the index of refraction n. For n larger than 1.2, such an approximation was within 3% of the exact value. When n is smaller than 1.2, the error is larger, especially for small values of t. On this basis, this approximation should be used for a refractive index larger than 1.2. Table 1 lists the values of these constants as a function of the refractive index.

It should be evident that when the refractive index becomes equal to unity the interface reflectance reduces to zero and the interface function becomes equivalent to the exponential integral $B_1(t) = E_1(t)$. The exponential approximation has been applied frequently to the exponential integral and can be represented by

$$E_1(t) = a\exp(-bt) \tag{18}$$

Two choices for a and b have been used in the literature. The first uses $a = b = 2$. This is the result of making the integral and the first moment of both the function and the approximation equivalent. The second choice uses $a = b = (3)^{1/2}$, which when used in the radiative transfer equation reduces to the Milne Eddington approximation. Both have been used in this study and appear to give directional emittance value within 10% of each other when the index of refraction is in the range between 1.2 and 2. The results presented in this paper use the first choice. It should be emphasized that the exponential approximation for the interface function $B_n(t)$ when $n > 1.2$, is much better than the approximation for the exponential integral $E_1(t)$ because it does not contain any singularities.

The substitutions of these approximations in Eq. (14) yield the following integral equation with a separable kernel:

$$\phi_\nu(\tau_\nu) = 1 + \frac{\omega_\nu}{2}\int_0^\infty \phi_\nu(t)[a\exp(-b|\tau_\nu - t|) + a\exp[-b(\tau_\nu + t)] - a_n\exp[-b_n(\tau_\nu + t)]]\, dt \tag{19}$$

Table 1 Constants for the exponential approximation to the interface function

n_ν	1.2	1.33	1.4	1.5	1.6	1.7	1.8	1.9	2.0
a_n	0.531	0.375	0.324	0.263	0.219	0.185	0.158	0.137	0.120
b_n	1.233	1.177	1.157	1.132	1.113	1.097	1.085	1.075	1.067

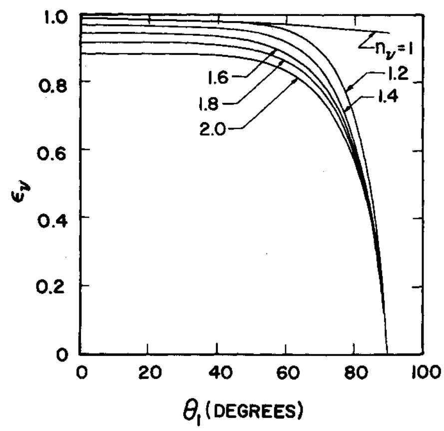

Fig. 2 Directional emittance for $\omega_\nu = 0.1$.

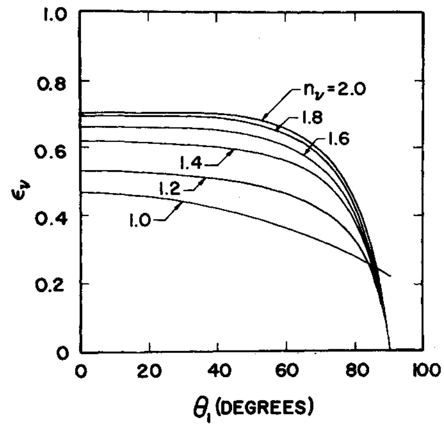

Fig. 4 Directional emittance for $\omega_\nu = 0.95$.

Solution

The above integral equation has been transformed to a differential equation by using a procedure common to an integral equation with a separable kernel. This transformation yields the following governing equation for the dimensionless source function:

$$\phi_\nu''' + b_n \phi_\nu'' - (b^2 - \omega_\nu ab)\phi_\nu' - b_n(b^2 - \omega_\nu ab)\phi_\nu = -b_n b^2 \tag{20}$$

for which the solution is given by

$$\phi_\nu(\tau_\nu) = C_1 \exp(-b_n \tau_\nu) + C_2 \exp[-(b^2 - \omega_\nu ab)^{1/2}\tau_\nu] + b/(b - \omega_\nu a) \tag{21}$$

The condition that the source function must be finite at infinity has been used to eliminate one of the constants. The remaining two constants can be evaluated by back substitution of Eq. (21) into the governing integral equation (19). By equating terms of equal power, the resulting expression for these constants is given by

$$\begin{aligned} C_1 = -\big[&4\omega_\nu b a_n b_n (b_n{}^2 - b^2)[b^2 - \omega_\nu ab]^{1/2} \times \\ &([b^2 - \omega_\nu ab]^{1/2} + b_n)\big] / \\ &\{2b_n(b - \omega_\nu a)\big[4b_n(b_n{}^2 - b^2)[b^2 - \omega_\nu ab]^{1/2} \times \\ &([b^2 - \omega_\nu ab]^{1/2} + b_n) + 4b_n\omega_\nu ab[b^2 - \omega_\nu ab]^{1/2} \times \\ &([b^2 - \omega_\nu ab]^{1/2} + b_n) + \omega_\nu a_n(b_n{}^2 - b^2) \times \\ &[b^2 - \omega_\nu ab]^{1/2}([b^2 - \omega_\nu ab]^{1/2} + b_n) + 2a_n b_n{}^2 \omega_\nu{}^2 ab\big]\} \end{aligned} \tag{22}$$

and

$$C_2 = \frac{b_n \omega_\nu ab C_1}{(b_n{}^2 - b^2)[b^2 - \omega_\nu ab]^{1/2}} \tag{23}$$

For the appropriate values of a, b, a_n, and b_n, the constants C_1 and C_2 can be determined as a function of the scattering albedo. In addition, the appropriate source function can be evaluated from Eq. (21). A closed form expression can also be deduced for the directional emittance from Eq. (12), and the resulting expression is given by

$$\begin{aligned} \varepsilon_\nu(\mu_1) = (1 - \omega_\nu)&\left[1 - \rho_\nu\left(\left[1 - \frac{(1-\mu_1{}^2)}{n_\nu{}^2}\right]^{1/2}\right)\right] \times \\ &\left\{\frac{C_1}{1 + b[1 - (1-\mu_1{}^2)/n_\nu{}^2]^{1/2}} + \right. \\ &\left.\frac{C_2}{1 + [b^2 - \omega_\nu ab]^{1/2}[1 - (1-\mu_1{}^2)/n_\nu{}^2]^{1/2}} + \frac{b}{b - \omega_\nu a}\right\} \end{aligned} \tag{24}$$

The special case of a unity refractive index has been treated separately and compared with the exact solution in an effort to evaluate the accuracy of the approximate method. Under this condition, the interface reflection is zero, and the interface function reduces to the exponential integral $B_1(t) = E_1(t)$. By following the same procedure as outlined above the dimensionless source function for this case becomes

$$\phi_\nu(\tau_\nu) = \frac{[(b^2 - \omega_\nu ab)^{1/2} - b]}{b - \omega_\nu a} \exp[-(b^2 - \omega_\nu ab)^{1/2}\tau_\nu] + \frac{b}{b - \omega_\nu a} \tag{25}$$

and the resulting directional emittance is given by

$$\varepsilon_\nu(\mu_1) = (1 - \omega_\nu)\left\{\frac{[(b^2 - \omega_\nu ab)^{1/2} - b]}{(b - \omega_\nu a)(\mu_1[b^2 - \omega_\nu ab]^{1/2} + 1)} + \frac{b}{b - \omega_\nu a}\right\} \tag{26}$$

The exact solution for the case of unity refractive index is given[9] by

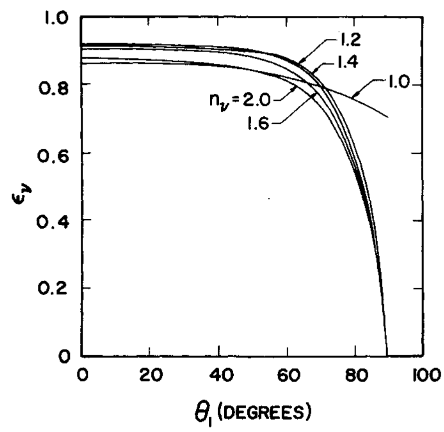

Fig. 3 Directional emittance for $\omega_\nu = 0.5$.

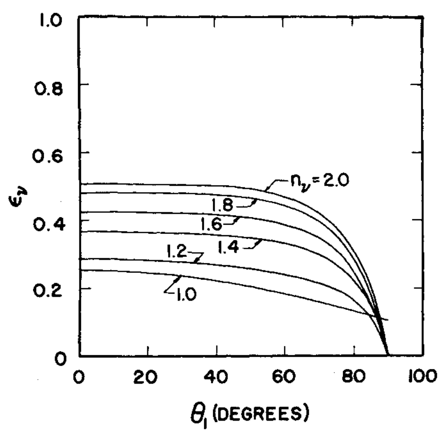

Fig. 5 Directional emittance for $\omega_\nu = 0.99$.

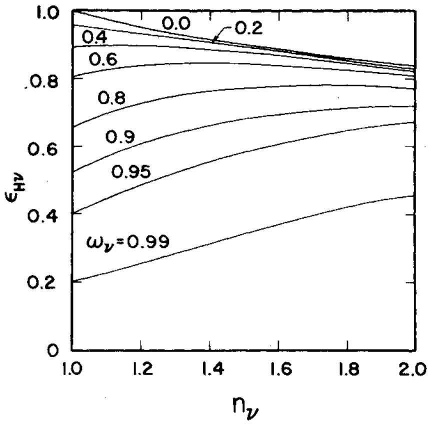

Fig. 6 Influence of refractive index and scattering albedo on hemispherical emittance.

$$\varepsilon_\nu(\mu_1) = (1-\omega_\nu)^{1/2} H(\mu_1) \tag{27}$$

where $H(\mu)$ is the Chandrasekhar H function. A comparison of the exact and the approximate solutions for the directional emittance shows an agreement within 5%. This agreement justifies the use of the method on the case when the index of refraction is different from unity.

Results and Discussion

The directional emittance has been presented for various optical conditions in Figs. 2–5. The figures exhibit clearly the dependence of this property on both the index of refraction and the scattering albedo. When the scattering albedo is small, e.g., $\omega_\nu = 0.1$ (Fig. 2), the directional emittance from a medium with an index of refraction larger than unity is lower than the one from a medium with an index of refraction of unity. This is due to the back reflection at the interface and the relatively low scattering contribution. For higher scattering albedo, e.g., 0.95, the reverse of the above is true (Fig. 4). The directional emittance from a medium with an index of refraction larger than unity is larger than the one from a medium with a unity index of refraction. The scattering mechanism in this case more than offsets the back reflection and results in a higher directional emittance. The case for $\omega_\nu = 0.5$ and $\omega_\nu = 0.99$ is presented in Figs. 3 and 5 to demonstrate further the behavior of this property. The large drop in the magnitude of the directional emittance beyond the 60° angle is due to the fact that the back reflectance increases rapidly near the critical angle and makes the interface transmittance approach zero.

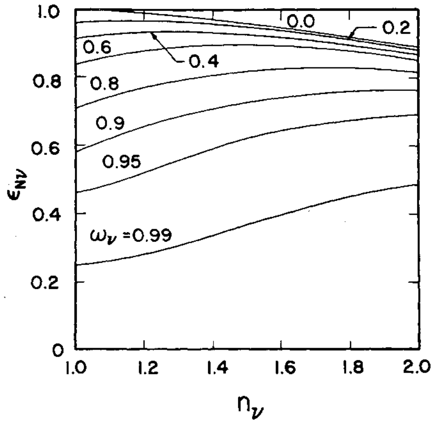

Fig. 7 Influence of refractive index and scattering albedo on normal emittance.

Table 2 Normal emittance

ω_ν	n_ν 1.0	1.2	1.33	1.4	1.5	1.6	1.7	1.8	1.9	2.0
0	1.0	0.992	0.980	0.972	0.960	0.947	0.933	0.918	0.904	0.889
0.1	0.982	0.981	0.972	0.965	0.954	0.942	0.929	0.915	0.901	0.886
0.2	0.962	0.969	0.963	0.957	0.947	0.936	0.923	0.911	0.897	0.883
0.3	0.939	0.955	0.951	0.947	0.939	0.929	0.917	0.906	0.892	0.879
0.4	0.911	0.937	0.937	0.935	0.928	0.920	0.910	0.899	0.886	0.874
0.5	0.879	0.914	0.919	0.919	0.915	0.908	0.900	0.890	0.879	0.867
0.6	0.838	0.884	0.895	0.897	0.896	0.892	0.886	0.879	0.868	0.857
0.7	0.784	0.842	0.859	0.865	0.869	0.868	0.865	0.861	0.852	0.843
0.8	0.708	0.776	0.803	0.814	0.824	0.828	0.830	0.831	0.825	0.819
0.85	0.655	0.726	0.759	0.773	0.788	0.796	0.802	0.805	0.803	0.798
0.9	0.581	0.653	0.693	0.711	0.731	0.745	0.755	0.765	0.765	0.764
0.925	0.531	0.600	0.644	0.664	0.688	0.705	0.719	0.732	0.735	0.737
0.95	0.463	0.527	0.573	0.596	0.624	0.645	0.664	0.681	0.688	0.693
0.975	0.360	0.411	0.456	0.481	0.512	0.537	0.562	0.586	0.598	0.609
0.99	0.250	0.283	0.322	0.344	0.374	0.399	0.426	0.454	0.470	0.485
1.00	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Table 3 Hemispherical emittance

ω_ν	n_ν 1.0	1.2	1.33	1.4	1.5	1.6	1.7	1.8	1.9	2.0
0	1.0	0.956	0.934	0.923	0.908	0.894	0.880	0.866	0.852	0.839
0.1	0.976	0.945	0.926	0.916	0.902	0.889	0.876	0.862	0.849	0.837
0.2	0.950	0.932	0.916	0.908	0.895	0.883	0.871	0.858	0.846	0.834
0.3	0.920	0.916	0.905	0.898	0.887	0.876	0.865	0.853	0.841	0.830
0.4	0.885	0.897	0.890	0.885	0.877	0.867	0.857	0.847	0.836	0.825
0.5	0.844	0.873	0.871	0.869	0.863	0.855	0.847	0.838	0.828	0.818
0.6	0.795	0.841	0.847	0.847	0.844	0.839	0.834	0.827	0.818	0.809
0.7	0.732	0.797	0.811	0.815	0.817	0.816	0.813	0.810	0.803	0.795
0.8	0.647	0.730	0.755	0.764	0.773	0.777	0.780	0.781	0.776	0.772
0.85	0.589	0.670	0.711	0.724	0.738	0.746	0.752	0.756	0.755	0.752
0.9	0.512	0.606	0.646	0.664	0.683	0.697	0.708	0.717	0.719	0.720
0.925	0.463	0.554	0.598	0.618	0.642	0.685	0.673	0.686	0.690	0.693
0.95	0.398	0.483	0.530	0.553	0.581	0.601	0.620	0.638	0.645	0.652
0.975	0.302	0.372	0.419	0.443	0.474	0.499	0.523	0.547	0.560	0.571
0.99	0.204	0.253	0.293	0.315	0.344	0.370	0.396	0.423	0.439	0.455
1.00	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

The hemispherical emittance is presented in Fig. 6 as a function of the scattering albedo for different values of the refractive index. The behavior of this property is similar, in trend, to the normal emittance; however, it is smaller in magnitude as can be seen from a comparison with Fig. 7. This feature is due to the angular distribution of the directional emittance, which is clearly not diffuse in nature. One should note that in the case of a metal the hemispherical emittance is larger than the normal emittance. The magnitudes for the normal and the hemispherical emittances are presented in Tables 2 and 3, respectively.

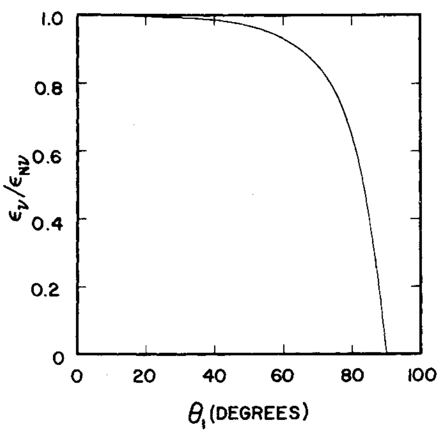

Fig. 8 Generalized graph for $1.2 < n_\nu < 2$ and $0 < \omega_\nu < 1$.

The curves for the directional emittance appear to be quite similar in nature. On this basis, a normalized curve is presented where the ratio of the directional emittance to the normal emittance is plotted vs angle in Fig. 8. It has been found that in the range between $n_v = 1.2$ and $n_v = 2$ this ratio is approximately independent of the refractive index and of the scattering albedo. Figure 8 can be used with Fig. 7 to approximate the directional emittance over a wide range of optical conditions (for a refractive index between 1.2 and 2 and a scattering albedo between 0 and 1). The generalized plot predicts the directional emittance in the above optical range to within 15%.

The accuracy of the approximate method has been examined by comparing it with the exact solution, which is available for a refractive index of unity, Eq. (27). The predicted values for the directional emittance from Eq. (26) are within 5% of the exact solution over the entire range of the scattering albedo. A further check has been attempted by extending the limited results of Turner and Love[5] to a larger optical depth to approach the semi-infinite case, which is presented in this study. Reasonable comparison can be made when the scattering albedo is 0.95; however, when the scattering albedo is 0.99, the slope of the curves are too small and the extrapolation to the infinite case is difficult. This is because the larger scattering albedo represents a relatively smaller absorption coefficient and thus requires a larger optical path to approach the thick or the semi-infinite limit.

Conclusions

The above discussion indicates clearly the important role that is played by the index of refraction and the scattering albedo in determining directional and hemispherical emittances. An increase in the refractive index can either increase or decrease the directional emittance depending on the magnitude of the scattering albedo. The close form expression for the directional emittance has the advantage of being relatively quick and easy to calculate. In addition it exhibits clearly the effective role of the various physical parameters. The generalized curve allows for a quick, but approximate calculation of the emittance for a wide range of optical parameters. The exponential kernel substitution, which has been used to obtain an approximate solution for the governing equations, is believed to predict the directional emittance to within 10% of the exact value.

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